Modeling and Fabrication with Specified Discrete Equivalence Classes - Supplement

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1 APPENDIX

1.1 Proposition 1

Prop. 1. The shape of $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ is independent of the translation $\mathbf{b}$, and so do C. Given any R, the best $\mathbf{b}$ minimizing $\|\xi(R)\|_\infty$ is the vector from the origin $\mathbf{o}$ to the center $\mathbf{c}$ of C, and the minimum $\|\xi(R)\|_\infty$ is the radius r_c of C.

Proof. Independence of the translation is deduced since the edge vector $\mathbf{u}_j\mathbf{u}_i = \mathbf{u}_i - \mathbf{u}_j = (\mathbf{v}_i - R\mathbf{p}_i - \mathbf{b}) - (\mathbf{v}_j - R\mathbf{p}_j - \mathbf{b}) = (\mathbf{v}_i - R\mathbf{p}_i) - (\mathbf{v}_j - R\mathbf{p}_j)$ is independent of the translation $\mathbf{b}$. Hence, the shape of $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ and the minimum covering circle (MCC) C of $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ are independent of the translation $\mathbf{b}$ (Fig. 1).

Given a rotation R, we get the rotated template triangle $\triangle \mathbf{p}'_0\mathbf{p}'_1\mathbf{p}'_2$, where $\mathbf{p}'_i = R\mathbf{p}_i$. $\triangle \mathbf{u}'_0\mathbf{u}'_1\mathbf{u}'_2$ is the triangle without the translation $\mathbf{b}$, i.e., $\mathbf{u}'_i = \mathbf{p}'_i - \mathbf{v}_i$. We denote $V_{\triangle \mathbf{u}'_0\mathbf{u}'_1\mathbf{u}'_2} = \{\mathbf{u}'_0, \mathbf{u}'_1, \mathbf{u}'_2\}$. Let $\mathbf{q} = \mathbf{b} + \mathbf{o}$. Define $d_S(\mathbf{v}) = \max_{\mathbf{s}\in S} \|\mathbf{v} - \mathbf{s}\|_2$ as the distance from a point $\mathbf{v}$ to a point set S. Then, we have:

$$
\begin{aligned}
\|\xi\|_\infty &= \max\{\|\mathbf{v}_0 - \mathbf{p}'_0 - \mathbf{b}\|_2, \|\mathbf{v}_1 - \mathbf{p}'_1 - \mathbf{b}\|_2, \|\mathbf{v}_2 - \mathbf{p}'_2 - \mathbf{b}\|_2\} \\
&= \max\{\|\mathbf{u}'_0 - \mathbf{b}\|_2, \|\mathbf{u}'_1 - \mathbf{b}\|_2, \|\mathbf{u}'_2 - \mathbf{b}\|_2\} \\
&= \max\{\|\mathbf{u}'_0 - \mathbf{q}\|_2, \|\mathbf{u}'_1 - \mathbf{q}\|_2, \|\mathbf{u}'_2 - \mathbf{q}\|_2\} \\
&= d_{V_{\triangle \mathbf{u}'_0\mathbf{u}'_1\mathbf{u}'_2}}(\mathbf{q}).
\end{aligned}
$$

The problem $\min \|\xi\|_\infty$ is converted to finding the best point $\mathbf{q}$ to minimize the distance from $\mathbf{q}$ to the vertices of $\triangle \mathbf{u}'_0\mathbf{u}'_1\mathbf{u}'_2$.

We claim that the best point $\mathbf{q}$ for minimizing $d_{V_{\triangle \mathbf{u}'_0\mathbf{u}'_1\mathbf{u}'_2}}(\mathbf{q})$ is the center $\mathbf{c}$ of C. Otherwise, there must be a point $\mathbf{o}^*$ s.t. $d_{V_{\triangle \mathbf{u}'_0\mathbf{u}'_1\mathbf{u}'_2}}(\mathbf{o}^*) < d_{V_{\triangle \mathbf{u}'_0\mathbf{u}'_1\mathbf{u}'_2}}(\mathbf{c}) = r_c$. Then, setting $r^* = d_{V_{\triangle \mathbf{u}'_0\mathbf{u}'_1\mathbf{u}'_2}}(\mathbf{o}^*)$, the circle $C_{\mathbf{o}^*}(r^*)$ covers all the vertices of $\triangle \mathbf{u}'_0\mathbf{u}'_1\mathbf{u}'_2$ and $r^* < r_c$, which contradicts the definition of MCC. As a result, $\mathbf{q} = \mathbf{c}$ and $\mathbf{b} = \mathbf{c} - \mathbf{o}$. Since the MCC of $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ is independent of the translation $\mathbf{b}$, $\min \|\xi\|_\infty = d_{V_{\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2}}(\mathbf{c}) = r_c$. □

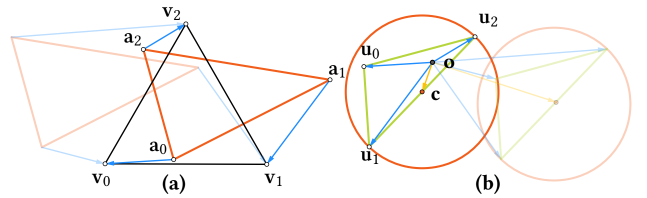

Fig. 1. Independent of the translation. (a) $\triangle \mathbf{v}_0\mathbf{v}_1\mathbf{v}_2$ is a triangle of the remeshed mesh $\mathcal{R}$ and $\triangle \mathbf{a}_0\mathbf{a}_1\mathbf{a}_2$ indicates the template triangle $\triangle \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$ after a rigid transformation $R, \mathbf{b}$. (b) Move the starting points of the three vectors $(\mathbf{v}_0 - \mathbf{a}_0, \mathbf{v}_1 - \mathbf{a}_1, \mathbf{v}_2 - \mathbf{a}_2)$ to the origin $\mathbf{o}$. The orange circle is the MCC of $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ and $\mathbf{c}$ is its center. Here $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ is an obtuse triangle and $\mathbf{c}$ is the midpoint of its longest edge. The transparent figures show another case with the same rotation but the different translation. The shape of green triangle $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ remains the same, so do the MCC.

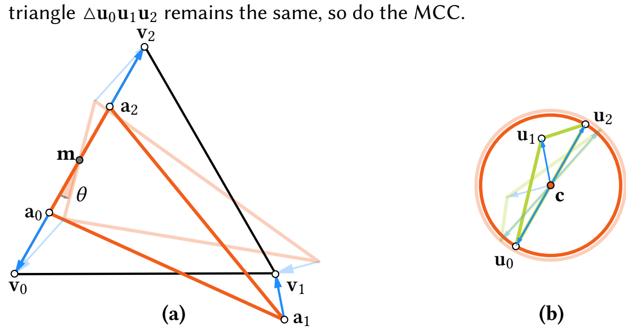

Fig. 2. The obtuse triangle case. (a) $\triangle \mathbf{v}_0\mathbf{v}_1\mathbf{v}_2$ is a triangle of $\mathcal{R}$ and $\triangle \mathbf{a}_0\mathbf{a}_1\mathbf{a}_2$ indicates the template triangle $\triangle \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$ after best rigid transformation $R, \mathbf{b}$. $\mathbf{a}_0\mathbf{a}_2$ and $\mathbf{v}_0\mathbf{v}_2$ coincide and their midpoints coincide at a point, denoted as $\mathbf{m}$. (b) Move the starting points of the three vectors $(\mathbf{v}_0 - \mathbf{a}_0, \mathbf{v}_1 - \mathbf{a}_1, \mathbf{v}_2 - \mathbf{a}_2)$ to the origin $\mathbf{o}$. The orange circle is the MCC of $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ and $\mathbf{c}$ is its center. $\mathbf{u}_0\mathbf{u}_2$ is the longest edge. Based on *Prop. 1*, the origin $\mathbf{o}$ is at the center $\mathbf{c}$ of C after the best translation $\mathbf{b}$. The transparent figures show another case with the best translation but another rotation. Corresponding edges $\mathbf{e}_f$ and $\mathbf{e}_t$ do not coincide and the radius of its MCC is bigger.

1.2 Proposition 2

Prop. 2. The longest edge of $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ corresponds to an edge of $\mathbf{f}$ (denoted as $\mathbf{e}_f$) and an edge of $\mathbf{t}$ (denoted as $\mathbf{e}_t$), respectively. Then, if $\triangle \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ is an obtuse triangle when $\|\xi\|_\infty$ reaches the minimum, then $\mathbf{e}_f$ and $\mathbf{e}_t$ coincide and their midpoints coincide.

Proof. Without the loss of generality, assume the longest edge is $\mathbf{u}_0\mathbf{u}_2$ and the best rotation is R^* when $\|\xi\|_\infty$ obtains the minimum. Considering MCC in the obtuse case, the radius r_c is the half of the longest side $\mathbf{u}_0\mathbf{u}_2$ and the center $\mathbf{c}$ is at the midpoint of $\mathbf{u}_0\mathbf{u}_2$. Based on *Prop. 1*, the origin $\mathbf{o}$ is at the center $\mathbf{c}$ of C after the best

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translation $\mathbf{b}$, indicating that $\overrightarrow{\mathbf{a}_0\mathbf{v}_0} = \overrightarrow{\mathbf{c}\mathbf{u}_0} = -\overrightarrow{\mathbf{c}\mathbf{u}_2} = -\overrightarrow{\mathbf{a}_2\mathbf{v}_2}$. Hence, the midpoints of $\mathbf{a}_0\mathbf{a}_2$ and $\mathbf{u}_0\mathbf{u}_2$ coincide at a point, denoted as $\mathbf{m}$ (Fig. 2).

Next, we prove that $\mathbf{e}_f$ and $\mathbf{e}_t$ coincide by contradiction. Since $\triangle\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ and C are independent of the translation, the center of rotation does not influence the shape of $\triangle\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ and C. Without the loss of generality, we now rotate $\triangle\mathbf{a}_0\mathbf{a}_1\mathbf{a}_2$ around $\mathbf{m}$. Suppose that $\mathbf{a}_0\mathbf{a}_2$ and $\mathbf{u}_0\mathbf{u}_2$ are not coincided after the best rotation R^*. Since $\triangle\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ is an obtuse triangle, we have $|\mathbf{a}_1\mathbf{v}_1| = |\mathbf{c}\mathbf{u}_1| < |\mathbf{c}\mathbf{u}_0| = |\mathbf{a}_0\mathbf{v}_0| = |\mathbf{a}_2\mathbf{v}_2| = r$. Thus, we can rotate $\triangle\mathbf{a}_0\mathbf{a}_1\mathbf{a}_2$ a small angle δ_θ to a new rotated template triangle, denoted as $\triangle\mathbf{a}_0'\mathbf{a}_1'\mathbf{a}_2'$, whose $|\mathbf{v}_1\mathbf{a}_1'|$ is still smaller than $|\mathbf{v}_0\mathbf{a}_0'| = |\mathbf{v}_2\mathbf{a}_2'| = r'$ and $r' < r$. However, $r' < r$ contradicts the assertion that rotation R^* is the best rotation. □

1.3 $f(\theta)$

Based on *Prop. 1*, the shape of $\triangle\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ and C are also independent of the center of rotation. In the acute triangle case, we place $\mathbf{a}_0$ at the origin $\mathbf{o}$, then the center of rotation R is $\mathbf{a}_0$. We draw the auxiliary lines, as shown in Fig. 3, where quadrilaterals $\mathbf{a}_0\mathbf{a}_1\mathbf{v}_1\mathbf{u}_1$ and $\mathbf{a}_0\mathbf{a}_2\mathbf{v}_2\mathbf{u}_2$ are parallelograms. Then, $\triangle\mathbf{v}_0\mathbf{u}_1\mathbf{u}_2$ is the same as $\triangle\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$. Thus, we only need to find the radius of the circumcircle of $\triangle\mathbf{v}_0\mathbf{u}_1\mathbf{u}_2$:

$$\min_R r_c = \min_R \frac{\|\mathbf{u}_1 - \mathbf{v}_0\|_2 \cdot \|\mathbf{u}_2 - \mathbf{u}_1\|_2 \cdot \|\mathbf{v}_0 - \mathbf{u}_2\|_2}{4\text{Area}(\triangle\mathbf{v}_0\mathbf{u}_1\mathbf{u}_2)},$$

where

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

The expression for r_c can be derived as:

$$\begin{aligned} f(\theta) = r_c &= \frac{\|\mathbf{u}_1 - \mathbf{v}_0\|_2 \cdot \|\mathbf{u}_2 - \mathbf{u}_1\|_2 \cdot \|\mathbf{v}_0 - \mathbf{u}_2\|_2}{4\text{Area}(\triangle\mathbf{v}_0\mathbf{u}_1\mathbf{u}_2)} \\ &= \sqrt{\frac{(\alpha_1 + \alpha_2\cos\theta + \alpha_3\sin\theta)(\alpha_4 + \alpha_5\cos\theta + \alpha_6\sin\theta)(\alpha_7 + \alpha_8\cos\theta + \alpha_9\sin\theta)}{(\alpha_{10} + \alpha_{11}\cos\theta + \alpha_{12}\sin\theta)^2}} \\ &= \sqrt{\frac{(\alpha_1 + \alpha_2\frac{1-t^2}{t^2+1} + \alpha_3\frac{2t}{t^2+1})(\alpha_4 + \alpha_5\frac{1-t^2}{t^2+1} + \alpha_6\frac{2t}{t^2+1})(\alpha_7 + \alpha_8\frac{1-t^2}{t^2+1} + \alpha_9\frac{2t}{t^2+1})}{(\alpha_{10} + \alpha_{11}\frac{1-t^2}{t^2+1} + \alpha_{12}\frac{2t}{t^2+1})^2}} \\ &= g(t), \end{aligned}$$

where $t = \tan(\theta/2)$ and $\alpha_1, ..., \alpha_{12}$ are the constants related to triangles $\mathbf{f} = \triangle\mathbf{v}_0\mathbf{v}_1\mathbf{v}_2$ and $\mathbf{t} = \triangle\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$,

$$\begin{aligned}
\alpha1 &= x_{v_0}^2 - 2x_{v_0}x_{v_1} + x_{v_1}^2 + x_{p_0}^2 - 2x_{p_0}x_{p_1} + x_{p_1}^2 + y_{v_0}^2 - 2y_{v_0}y_{v_1} + y_{v_1}^2 + y_{p_0}^2 - 2y_{p_0}y_{p_1} + y_{p_1}^2,\\
\alpha2 &= 2\,(x_{v_1}(x_{p_0} - x_{p_1}) + x_{v_0}(-x_{p_0} + x_{p_1}) - (y_{v_0} - y_{v_1})(y_{p_0} - y_{p_1})),\\
\alpha3 &= 2\,(x_{p_1}(y_{v_0} - y_{v_1}) + x_{p_0}(-y_{v_0} + y_{v_1}) + (x_{v_0} - x_{v_1})(y_{p_0} - y_{p_1})),\\
\alpha4 &= x_{v_0}^2 - 2x_{v_0}x_{v_2} + x_{v_2}^2 + x_{p_0}^2 - 2x_{p_0}x_{p_2} + x_{p_2}^2 + y_{v_0}^2 - 2y_{v_0}y_{v_2} + y_{v_2}^2 + y_{p_0}^2 - 2y_{p_0}y_{p_2} + y_{p_2}^2,\\
\alpha5 &= 2\,(x_{v_2}(x_{p_0} - x_{p_2}) + x_{v_0}(-x_{p_0} + x_{p_2}) - (y_{v_0} - y_{v_2})(y_{p_0} - y_{p_2})),\\
\alpha6 &= 2\,(x_{p_2}(y_{v_0} - y_{v_2}) + x_{p_0}(-y_{v_0} + y_{v_2}) + (x_{v_0} - x_{v_2})(y_{p_0} - y_{p_2})),\\
\alpha7 &= x_{v_1}^2 - 2x_{v_1}x_{v_2} + x_{v_2}^2 + x_{p_1}^2 - 2x_{p_1}x_{p_2} + x_{p_2}^2 + y_{v_1}^2 - 2y_{v_1}y_{v_2} + y_{v_2}^2 + y_{p_1}^2 - 2y_{p_1}y_{p_2} + y_{p_2}^2,\\
\alpha8 &= 2\,(x_{v_2}(x_{p_1} - x_{p_2}) + x_{v_1}(-x_{p_1} + x_{p_2}) - (y_{v_1} - y_{v_2})(y_{p_1} - y_{p_2})),\\
\alpha9 &= 2\,(x_{p_2}(y_{v_1} - y_{v_2}) + x_{p_1}(-y_{v_1} + y_{v_2}) + (x_{v_1} - x_{v_2})(y_{p_1} - y_{p_2})),\\
\alpha10 &= 2\,(x_{v_1}y_{v_0} - x_{v_2}y_{v_0} - x_{v_0}y_{v_1} + x_{v_2}y_{v_1} + x_{v_0}y_{v_2} - x_{v_1}y_{v_2} + x_{p_1}y_{p_0} - x_{p_2}y_{p_0} - x_{p_0}y_{p_1}\\
&\quad + x_{p_2}y_{p_1} + x_{p_0}y_{p_2} - x_{p_1}y_{p_2}),\\
\alpha11 &= 2\,(x_{p_2}y_{v_0} + x_{p_0}y_{v_1} - x_{p_2}y_{v_1} - x_{p_0}y_{v_2} - x_{p_1}y_{v_0} + x_{p_1}y_{v_2} - x_{v_1}y_{p_0} + x_{v_2}y_{p_0} + x_{v_0}y_{p_1}\\
&\quad - x_{v_2}y_{p_1} - x_{v_0}y_{p_2} + x_{v_1}y_{p_2}),\\
\alpha12 &= 2\,(-x_{v_1}x_{p_0} + x_{v_2}x_{p_0} + x_{v_0}x_{p_1} - x_{v_2}x_{p_1} - x_{v_0}x_{p_2} + x_{v_1}x_{p_2} - y_{v_1}y_{p_0} + y_{v_2}y_{p_0} + y_{v_0}y_{p_1}\\
&\quad - y_{v_2}y_{p_1} - y_{v_0}y_{p_2} + y_{v_1}y_{p_2}).
\end{aligned}$$

Since $g(t) = f(\theta) \geq 0$, $\arg\min_t g(t) = \arg\min_t G(t) = g^2(t)$. To solve $\min_t G(t)$, we differentiate $G(t)$ and take the numerator of $G'(t)$ as $P(t)$. Since $P(t)$ is a tenth degree polynomial, we use Jenkins-Traub algorithm [Jenkins and Traub 1970] to find its all ten roots. We first compare the function values at the ten roots to find the minimum and then compute the corresponding θ as the result.

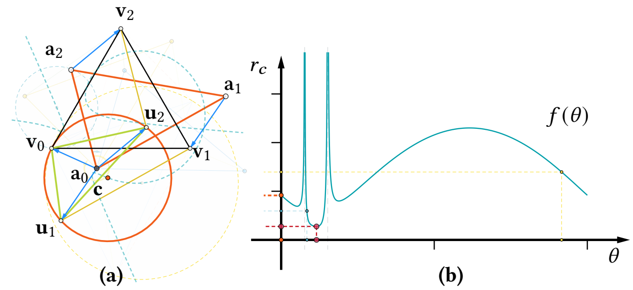

Fig. 3. The acute triangle case. (a) $\triangle\mathbf{v}_0\mathbf{v}_1\mathbf{v}_2$ is a triangle of $\mathcal{R}$ and $\triangle\mathbf{a}_0\mathbf{a}_1\mathbf{a}_2$ indicates the template triangle $\triangle\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$ after a rigid transformation $R, \mathbf{b}$. Quadrilaterals $\mathbf{a}_0\mathbf{a}_1\mathbf{v}_1\mathbf{u}_1$ and $\mathbf{a}_0\mathbf{a}_2\mathbf{v}_2\mathbf{u}_2$ are parallelograms. $\mathbf{c}$ is the center of the circumcircle of $\triangle\mathbf{a}_0\mathbf{u}_1\mathbf{u}_2$, which is the same as the MCC of $\triangle\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2$ for the acute triangle. (b) The image of the function $f(\theta) = r_c$.

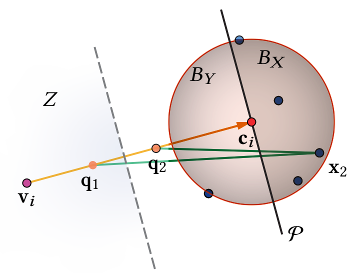

Fig. 4. Blue points in the figure are the templates' vertices corresponding to $\mathbf{v}_i$, i.e., points in $S(\mathbf{v}_i)$. The orange circle represents the minimum bounding sphere of $S(\mathbf{v}_i)$, denoted as $B_{S(\mathbf{v}_i)}$. $\mathbf{c}_i$ is the center of $B_{S(\mathbf{v}_i)}$. $\mathcal{P}$ is the plane perpendicular to the line $\overline{\mathbf{v}_i\mathbf{c}_i}$ through the point $\mathbf{c}_i$ divideing the minimum bounding sphere $B_{S(\mathbf{v}_i)}$ into two parts B_X and B_Y. The bisecting plane of the edge $\mathbf{q}_1\mathbf{q}_2$ divide the space into two parts, and we denote Z as the part containing $\mathbf{q}_1$. When moving $\mathbf{v}_i$ from its start point to $\mathbf{c}_i$, $\max_{s\in S(\mathbf{v}_i)} \|s - (\mathbf{v}_i)\|$ decreases monotonically and so does the $\max_{\mathbf{f}\in\Omega_i} d_{\text{assembly}}(\mathbf{f})$.

1.4 Proposition 3

Prop. 3. For each vertex $\mathbf{v}_i$, the maximum assembly error on its one-ring triangles (denoted as Ω_i) of $\mathbf{v}_i$ is:

$$d(\alpha_i) = \max_{\mathbf{f}\in\Omega_i} d_{\text{assembly}}(\mathbf{f}) = \max_{\mathbf{f}\in\Omega_i} \min_{\substack{\mathbf{t}\in\mathcal{T} \\ j\in\{1,\cdots,6\}}} d_{\max}(\mathbf{f}, \mathbf{t}^{\phi_j}) \quad (1)$$

where $0 \leq \alpha_i \leq 1$ is the step size. Then, $d(\alpha_i)$ monotonically decreases with respect to α_i.

Proof. Let $\mathcal{P}$ be the plane perpendicular to the line $\overline{\mathbf{v}_i\mathbf{c}_i}$ through the point $\mathbf{c}_i$ (Fig. 4). $S(\mathbf{v}_i)$ consists of the templates' vertices corresponding to $\mathbf{v}_i$. $\mathcal{P}$ divides the minimum bounding sphere $B_{S(\mathbf{v}_i)}$ into two parts B_X and B_Y. Let B_X be the part far away from $\mathbf{v}_i$ and $B_X' = B_X \cup \mathcal{P}$. Then, the point set $X = S(\mathbf{v}_i) \cap B_X'$ are not empty; otherwise, there is a bounding sphere having a smaller radius than $B_{S(\mathbf{v}_i)}$, which contradicts the assertion that $B_{S(\mathbf{v}_i)}$ is the minimum bounding sphere. Let $\mathbf{v}_i' = \mathbf{v}_i + \alpha_i\mathbf{d}$, we define:

$$\delta(\alpha_i) = d_{S(\mathbf{v}_i)}(\mathbf{v}_i') = \max_{s\in S(\mathbf{v}_i)} \|s - (\mathbf{v}_i + \alpha_i\mathbf{d})\|, \quad (2)$$

where $d_S(\mathbf{v}) = \max_{\mathbf{s}\in S} \|\mathbf{v}-\mathbf{s}\|_2$ is defined as the distance from a point $\mathbf{v}$ to a point set S.

$\delta(\alpha_i)$ is monotonically decreasing, and we prove it by contradiction. According to the fact that $\delta(\alpha_i) = d_S(\mathbf{v}'_i) = d_X(\mathbf{v}'_i) \geq d_Y(\mathbf{v}'_i)$, we only need to focus on the set X and $d_X(\mathbf{v}'_i)$. Suppose that $\delta(\alpha_i), \alpha_i \in [0,1]$ does not monotonically decrease, then there are two different values $\alpha_i^1 < \alpha_i^2$, s.t. $\delta(\alpha_i^1) < \delta(\alpha_i^2)$. Namely, $\exists\, \mathbf{q}_1, \mathbf{q}_2 \in \overline{\mathbf{v}_i\mathbf{c}_i}, \mathbf{q}_1 = \mathbf{v}_i + \alpha_i^1\mathbf{d}, \mathbf{q}_2 = \mathbf{v}_i + \alpha_i^2\mathbf{d}$ and $\alpha_i^1 < \alpha_i^2$, s.t. $d_X(\mathbf{q}_1) < d_X(\mathbf{q}_2)$. Let $\mathbf{x}_2 \in X$ be the point where $d_X(\mathbf{q}_2)$ is obtained, i.e., $d_X(\mathbf{q}_2) = \|\mathbf{q}_2 - \mathbf{x}_2\|$. Then, $\|\mathbf{q}_1 - \mathbf{x}_2\| \leq d_X(\mathbf{q}_1) < d_X(\mathbf{q}_2) = \|\mathbf{q}_2 - \mathbf{x}_2\|$. The bisecting plane of the edge $\mathbf{q}_1\mathbf{q}_2$ divide the space into two parts, and we denote Z as the part containing $\mathbf{q}_1$. We have $Z \cap X = \emptyset$, and since $\mathbf{x}_2 \in X$, $\mathbf{x}_2 \notin Z$. Thus, $\|\mathbf{q}_1 - \mathbf{x}_2\| \geq \|\mathbf{q}_2 - \mathbf{x}_2\|$, and the contradiction arises.

Then, $\forall \alpha_i \in [0,1]$, we have:

$$d(\alpha_i) = \max_{\mathbf{f}\in\Omega_i} d_{\text{assembly}}(\mathbf{f}) = \max_{\mathbf{f}\in\Omega_i} \min_{\substack{\mathbf{t}\in\mathcal{T} \\ j\in\{1,\cdots,6\}}} d_{\max}(\mathbf{f}, \mathbf{t}^{\phi_j}) \geq \delta(\alpha_i). \quad (3)$$

$d(\alpha_i) = \max_{\mathbf{f}\in\Omega_i} d_{\text{assembly}}(\mathbf{f})$ must be obtained at one vertex on certain triangle $\mathbf{f} \in \Omega_i$, denoted as $\mathbf{v}^*$. If $\mathbf{v}^* = \mathbf{v}'_i$, then $d(\alpha_i) = \delta(\alpha_i)$; otherwise, $d(\alpha_i)$ is obtained at another point, indicating that $d(\alpha_i)$ does not change as α_i updates. More specifically, if $\exists \alpha'_i$, s.t. $d(\alpha'_i) > \delta(\alpha'_i)$, then $d(\alpha_i)$ is constant $\forall \alpha_i \in (\alpha'_i, 1)$; otherwise $d(\alpha_i) = \delta(\alpha_i)$. Hence, since $\delta(\alpha_i)$ is monotonically decreasing, $d(\alpha_i)$ is also monotonically decreasing.

□

1.5 Proposition 4

Prop. 4. Given two triangles $A = \triangle \mathbf{a}_o\mathbf{a}_1\mathbf{a}_2$ and $B = \triangle \mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$, let $d_H(A,B)$ be the two-sided Hausdorff distance between two triangles and $\xi = (\|\mathbf{b}_0 - \mathbf{a}_0\|_2, \|\mathbf{b}_1 - \mathbf{a}_1\|_2, \|\mathbf{b}_2 - \mathbf{a}_2\|_2)$. Then,

$$d_H(A,B) \leq \|\xi\|_\infty \leq \|\xi\|_2.$$

PROOF. Since $\forall i \in \{0,1,2\}, d_H(A,B) \leq \|\mathbf{b}_i - \mathbf{a}_i\|_2 \leq \|\xi\|_2, d_H(A,B) \leq \|\xi\|_\infty \leq \|\xi\|_2$. □

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