MPM

Tuesday, February 1, 2022 11:03 AM

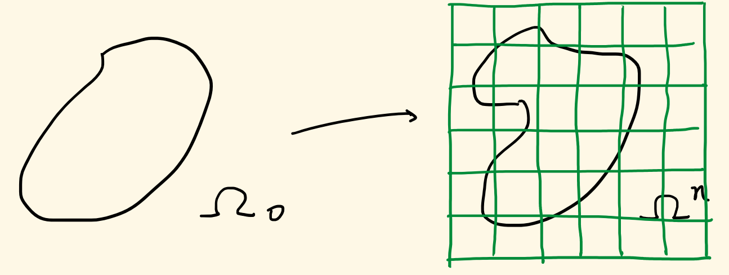
PDE: $R^0 \frac{\partial \underline{V}}{\partial t} = \nabla^{\underline{X}} \cdot \underline{P} + R^0 \underline{g}$, $\underline{X} \in \Omega^0, t \in (0,T)$

where , $\underline{P}$ = first Piola-krchoff stress

$= J \underline{\sigma} \underline{F}^{-T}$ ($\underline{\sigma}$: Cauchy stress , $J = \det(\underline{F})$, $\underline{F} = \frac{\partial \phi}{\partial \underline{X}}$)

FEM

- $t > t^n$, assume we know $\underline{\phi} : \Omega^0 \times [0, t^n]$
- interploting function



- chose interploting function N_i over grid. ($N_i : \Omega^n \to R$)
  - e.g. B-spline (Q1. How?) (on grid)
- define : $\hat{N}_i(\underline{X}) = N_i \circ \underline{\phi}(\underline{X}, t^n)$ ($\hat{N}_i : \Omega^0 \to R$

Then, do FEM in standard way.

$\underline{\phi}(\underline{X}, t) = \sum_i \underline{x}_i(t) \hat{N}_i(\underline{X})$, $t > t^n$

⇒ weak form (discrete)

$$\int_{\Omega_0} (\underline{w}_i \hat{N}_i)^T R^0 \frac{\partial^2 \phi}{\partial t^2} d\underline{X} = \int_{\Omega^0} (\underline{w}_i \hat{N}_i)^T \nabla^{\underline{X}} \cdot P \, d\underline{X} + \int_{\Omega^0} (\underline{w}_i \hat{N}_i)^T R^0 \underline{g} \, d\underline{X}$$

⇓

$$w_{i\alpha} \int_{\Omega^0} \hat{N}_i R^0 \underbrace{\hat{N}_j}_{\delta_{\alpha\beta}} d\underline{X} \cdot \frac{\partial^2 x_{j\beta}}{\partial t^2} = w_{i\alpha} \int_{\Omega^0} \hat{N}_i \frac{\partial P_{\alpha\beta}}{\partial X_\beta} d\underline{X} + w_{i\alpha} \int_{\Omega^0} R_0 \hat{N}_i g_\alpha d\underline{X}$$

or: $$w_{i\alpha} \underbrace{M_{i\alpha j\beta}}_{①} \frac{\partial^2 x_{j\beta}}{\partial t^2} = w_{i\alpha} \int_{\Omega^0} \frac{\partial}{\partial X_\beta}(\hat{N}_{i\alpha} P_{\alpha\beta}) - P_{\alpha\beta} \frac{\partial \hat{N}_{i\alpha}}{\partial X_\beta} d\underline{X} + w_{i\alpha} \int_{\Omega^0} R_0 \hat{N}_{i\alpha} g_\alpha d\underline{X}$$

$$= w_{i\alpha} \left(\int_{\partial\Omega^0} \hat{N}_{i\alpha} P_{\alpha\beta} N_\beta \, dS - \underbrace{\int_{\Omega^0} P_{\alpha\beta} \frac{\partial \hat{N}_i}{\partial X_\beta} d\underline{X}}_{②} \right) + w_{i\alpha} \int_{\Omega^0} R_0 \hat{N}_i g_\alpha d\underline{X}$$

(Q2. what's this?, is this the normal vector or the interplot function ?) (normal)

Now let's change variable and integrate over Ω^{t_n}

①. $M_{i\alpha j\beta} = \int_{\Omega^0} \hat{N}_i R^0 \hat{N}_j \, d\underline{X} \, \delta_{\alpha\beta}$

$= \int_{\Omega^{t^n}} N_i R(\phi^{-1}(\underline{x}, t^n), t^0) N_j \, 1/J^n \, d\underline{x} \cdot \delta_{\alpha\beta}$

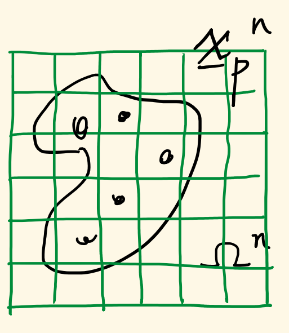
$= \int_{\Omega^{t^n}} N_i \rho(\underline{x}, t^n) N_j \, d\underline{x}$ (where $\rho(\underline{x}, t^n) = R(\underline{X}, 0) / J(\underline{X}, t^n)$)

② $\int_{\Omega^0} P_{\alpha\beta} \frac{\partial \hat{N}_i}{\partial X_\beta} d\underline{X}$

$= \int_{\Omega^{t^n}} P_{\alpha\beta} \frac{\partial \hat{N}_i}{\partial X_\beta} \circ (\underline{\phi}^{-1}(\underline{x}, t^n)) \, 1/J^n \, d\underline{x}$

$= \int_{\Omega^{t^n}} P_{\alpha\beta} \frac{\partial \hat{N}_i}{\partial x_\gamma} \underbrace{f_{\gamma\beta}(\underline{x}, t^n)}_{= F_{\gamma\beta}(\underline{X}, t^n)} \, 1/J^n \, d\underline{x}$

Introduce quadratic



①. $M_{i\alpha j\beta} \approx \sum_p N_i(\underline{x}_p) \rho(\underline{x}_p, t^n) N_j(\underline{x}_p) V_p^n$

$= \sum_p N_i(\underline{x}_p) m_p^n N_j(\underline{x}_p)$

($V_p^n = J_p^n V_p^0$, $m_p^n = R^n(\underline{X}_p) V_p^n$)

②. $\int_{\Omega^0} P_{\alpha\beta} \frac{\partial N_i}{\partial x_\gamma} f_{\gamma\beta} \, 1/J^n \, d\underline{x}$

$\approx \sum_p P_{\alpha\beta}(\underline{X}_p, t) \frac{\partial N_i}{\partial x_\gamma}(\underline{x}_p^n) F_{\gamma\beta}(\underline{X}_p, t^n) \underbrace{1/J_p^n V_p^n}_{V_p^0}$

e.g. Hyperelastic:

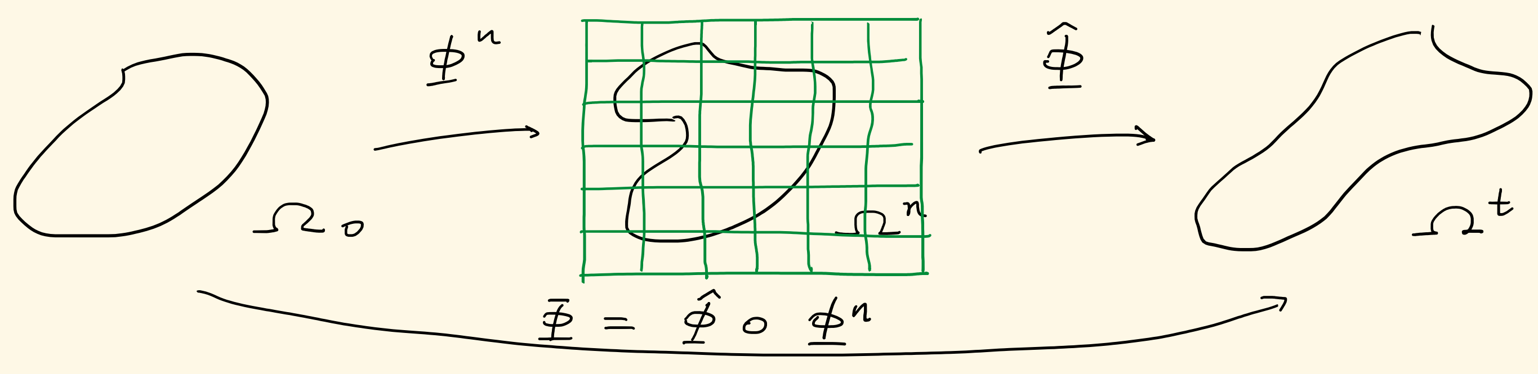
$P_{\alpha\beta}(\underline{X}_p, t) = \frac{\partial \psi}{\partial F_{\alpha\beta}}(\underline{F}(\underline{X}_p, t))$

(since $\underline{\phi}(\underline{X}, t) = \sum_i \hat{x}_i(t) N_i(\underline{\phi}(\underline{X}, t^n))$

$= \hat{\underline{\phi}}(\underline{\phi}(\underline{X}, t^n), t)$

we have : $\frac{\partial \underline{\phi}}{\partial \underline{X}} = \frac{\partial \hat{\underline{\phi}}}{\partial \underline{x}}(\underline{\phi}(\underline{X}, t^n), t) \cdot \underline{F}(\underline{X}, t^n)$

$= \hat{\underline{F}} \cdot \underline{F}^n$)



$\Rightarrow \underline{P}(\underline{X}_p, t) = \frac{\partial \psi}{\partial \underline{F}}\left(\left[\sum_j \hat{\underline{\phi}}_j(t) \frac{\partial N_j}{\partial \underline{x}}(\underline{x}_p^n) \right] \underline{F}_p^n \right)$

Thus,

$\cancel{R^0 \frac{\partial^2 \underline{\phi}}{\partial t^2}} = \nabla^{\underline{X}} \cdot \underline{P} + R^0 \underline{g}$ (for caudy static problem)

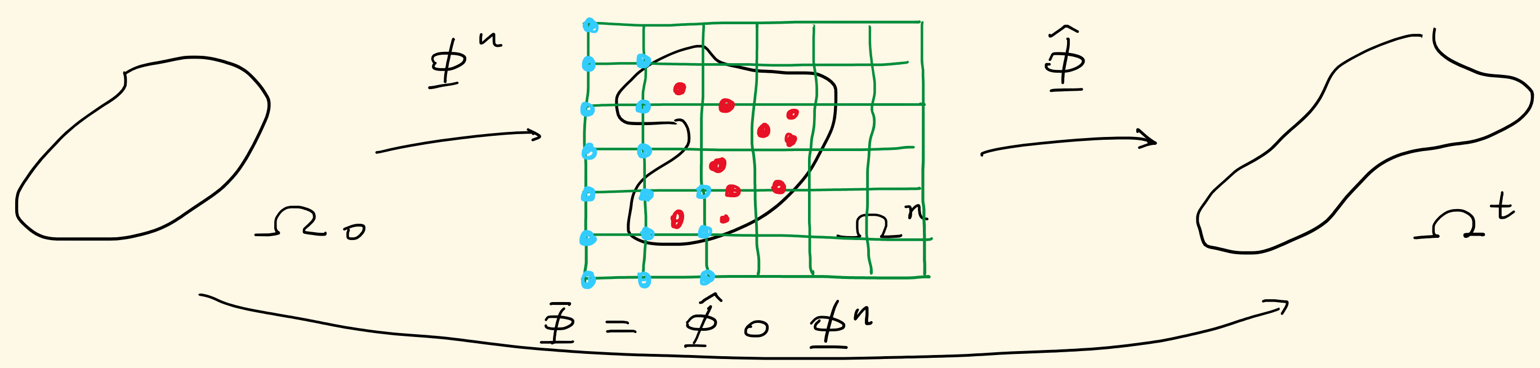
(Q3 : I forget what is this line stands for ...)

Now We have:

$\underline{\Phi}(\underline{x}, t) = \sum_i \hat{\underline{\Phi}}_i(t) N_i \circ \underline{\phi}(\underline{x}, t^n)$

$\frac{\partial \underline{\phi}}{\partial t}(\underline{x}, t) = \sum_i \frac{\partial \hat{\underline{\Phi}}_i(t)}{\partial t} N_i \circ \underline{\phi}(\underline{x}, t^n)$

$M_{i\alpha j\beta} \frac{\partial^2 \phi_{j\beta}}{\partial t^2}(t) = -\sum_p P_{\alpha\beta}(\underline{X}_p, t) \frac{\partial N_i}{\partial x_\gamma}(\underline{x}_p^n) F_{\gamma\beta}(\underline{X}_p, t^n) V_p^0 + \sum_p m_p N_i(\underline{x}_p^n) g_\alpha$



We use ∴ to have ∴,

$M_{i\alpha j\beta} = \sum_p m_p N_i(\underline{x}_p^n) N_j(\underline{x}_p^n) \delta_{\alpha\beta}$.

Do backward Euler

$$M_{i\alpha j\beta} \left(\frac{\frac{\hat{\phi}_{j\beta}^{n+1} - \hat{\phi}_{j\beta}^{n}}{\Delta t} - v^n_{j\beta}}{\Delta t} \right) = -\sum_p P_{\alpha\beta}\left(\hat{\underline{\phi}}_j^{n+1} \cdot \frac{\partial N_j}{\partial \underline{X}}(x_p^n)\right) F^n_{p\gamma\beta} \frac{\partial N_i}{\partial x_\gamma}(\underline{x}_p^n) V_p^0 + \sum_p m_p N_i(\underline{x}_p^n) g_\alpha$$

* Note: $\hat{\underline{\phi}}_j(t^n) = \hat{\underline{\phi}}_j = \underline{x}_j$ ⇐ "location of j^{th} grid node"

e.g. when $t = t^n$, $\hat{\underline{\Phi}} = \underline{I}$

- at last, we need $\underline{v}_j^n$.

  (around we have $\underline{v}_p^n$ on portfide)

  $\underline{v}_j^n \approx \sum_p m_p \underline{v}_p^n N_j(\underline{x}_p^n) / m_j^n$ ← grid node mass.

  ($m_j^n = \sum_p m_p N_j(\underline{x}_p^n)$)

Solve with Newton method, etc. We'll get $\hat{\underline{\Phi}}_j^{n+1}$.

then we know $\hat{\underline{\Phi}}(\underline{x}, t^{n+1}) = \hat{\underline{\Phi}}_j^{n+1} N_j(\underline{x})$

and let's interpolate motion to the perticals.

$\underline{x}_p^{n+1} = \underline{\phi}_j^{n+1} N_j(\underline{x}_p^n)$

$\underline{v}_p^{n+1} = \left(\frac{\underline{\Phi}_j^{n+1} - \underline{x}_j}{\Delta t} \right) N_j(\underline{x}_p^n)$

$= \frac{\underline{x}_p^{n+1} - \underline{x}_p}{\Delta t}$ # less angular momotum.

$\underline{F}_p^{n+1} = \hat{\underline{\phi}}_j^{n+1} \frac{\partial N_j}{\partial \underline{X}}(\underline{x}_p^n) \cdot \underline{F}_p^n$

↙ FLIP

$\underline{v}_p^n = \left(\frac{\underline{\Phi}_j^{n+1} - \underline{\Phi}_j^n}{\Delta t} - \underline{v}_j^n \right) N_j(\underline{x}_p^n) + \underline{v}_p^n$

this can greatly reduce the angular momtum loss.